PHYSICAL REVIEW E VOLUME 56, NUMBER 2 AUGUST 1997

Scattering matrices with block symmetries

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(Received 23 December 1996)

Scattering matrices with block symmetry, which correspond to a scattering process in cavities with geometrical symmetry, are analyzed. The distribution of the transmission coefficient is computed for a different number of channels in the case of a system with or without the time-reversal invariance. An interpolating formula for the case of gradual time reversal symmetry breaking is proposed. [S1063-651X(97)02007-2]

PACS number(s): 05.45.+b

I. INTRODUCTION

Ensembles of random matrices introduced in context of the theory of nuclear spectra by Dyson [1] a long time ago found a novel application in problems of chaotic scattering [2–8]. The S matrix corresponding to time reversal invariant problems can be described by matrices of circular orthogonal ensemble (COE), while the circular unitary ensemble (CUE) is applicable if no antiunitary symmetry exists.

The process of chaotic scattering in cavities with a geometrical symmetry can be represented by an S matrix with block symmetry. Such matrices were recently introduced by Gopar *et al.* [9], who discussed both cases: with and without time-reversal symmetry. They found a link between the invariant measure of such ensembles and the measures of canonical circular ensembles and computed the distribution of the transmission coefficient for one and two incoming channels.

In this Brief Report we generalize the results of Gopar *et al.* [9]. Using an idea of composed ensembles of random matrices we simplify analytical calculation of the transmission coefficient T. This method allows us to obtain the distributions $P(T)$ for interpolating ensembles corresponding to breaking of the time-reversal symmetry and to treat the case of large number of channels. Generating random unitary matrices of interpolating ensembles according to the method described in [10], we verify numerically proposed distributions.

II. BLOCK SYMMETRIC SCATTERING MATRICES

We introduce the random S matrices with block symmetry following the method of Gopar *et al.* [9]. The $2M\times 2M$ scattering matrix possess the structure

$$S=\begin{pmatrix} r & t \\ t & r \end{pmatrix}, \tag{2.1}$$

where $M\times M$ matrices r and t describe reflection and transmission processes, respectively. S matrix can be brought into a block diagonal form

$$S'=R_0 S R_0^T=\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}, \tag{2.2}$$

where $s_1=r+t$ and $s_2=r-t$ are unitary matrices and R_0 stands for rotation matrix consisting of M-dimensional unit matrices 1_M

$$R_0=\frac{1}{\sqrt{2}}\begin{pmatrix} 1_M & 1_M \\ -1_M & 1_M \end{pmatrix}. \tag{2.3}$$

Of great physical importance is the total transmission coefficient

$$T=\mathrm{tr}(tt^\dagger), \tag{2.4}$$

which is proportional to the conductance G of the cavity, $G=(2e^2/h)T$. Since $t=(s_1-s_2)/2$, the transmission coefficient reads

$$T=\frac{M}{2}-\frac{1}{2}\mathrm{Re}[\mathrm{tr}(s_1 s_2^\dagger)]. \tag{2.5}$$

Following Gopar *et al.* [9] we assume that unitary matrices s_1 and s_2 are statistically independent and pertain to the same universality class. For canonical circular ensembles of random matrices the joint probability distribution (JPD) of eigenphases is given by a single formula [11]

$$P_{U_\beta}[\varphi_1,\ldots,\varphi_M]=C_{\beta,M}\prod_{i>j}|e^{i\varphi_i}-e^{i\varphi_j}|^\beta, \tag{2.6}$$

where $C_{\beta,M}$ stands for normalization constant, while U_β represents Poissonian, orthogonal, and unitary circular ensemble for β equal to 0, 1, and 2, respectively.

Formula (2.5) contains a product of two unitary matrices $s_1 s_2$ (one can redefine the second matrix writing $s_2'=s_2^\dagger$). Further calculation are based on the relation linking the JPD of eigenvalues of the product of two unitary matrices drown independently from any canonical ensemble

$$P_{U_\beta\times U_\beta}[\varphi_1,\ldots,\varphi_M]=P_{U_\beta}[\varphi_1,\ldots,\varphi_M]. \tag{2.7}$$

The left hand side of the above equation formally represents the JPD characteristic of spectra of a composed ensemble defined via the product of two random matrices, each specified by a certain probability distribution. For the unitary ensemble ($\beta=2$) this equality follows from the invariance properties of CUE, which corresponds to the Haar measure on the unitary group. The same concerns the case $\beta=0$,

1063-651X/97/56(2)/2257(4)/$10.00

since the circular Poissonian ensemble (CPE) can be defined by the Haar measure in the space of diagonal unitary matrices. In the case of symmetric unitary matrices ($\beta=1$) this property follows from the definition of COE, which is invariant with respect to transformations $s \to s' = XsX^T$ [1], where X denotes any unitary matrix. The analyzed matrix $s_1 s_2$ is similar to $s_1^{1/2} s_2 s_1^{1/2}$. Since s_1 is symmetric, so is $s_1^{1/2}$, which may play the role of X in the invariance condition. The JPD of eigenvalues of $s_1 s_2$ (averaged over the composed ensemble) is thus the same as that of s_2 and is given by Eq. (2.6) with $\beta=1$. In spite of this result, the measure of the composed ensemble containing products of two symmetric unitary matrices differs form this characteristic of COE [12].

Since the JPD of eigenvalues does not define the entire probability distribution of an ensemble of random matrices, the formula (2.7) looses its meaning for noninteger values of the parameter β, characteristic to the transition between the canonical ensembles. However, in order to get possible interpolating formulas for the distribution of transmission coefficients, we will eventually allow β to take any real value in $[0,2]$.

Relation (2.7) allows one to write the transmission coefficient (2.5) in a simplified form

$$T = \frac{M}{2} - \frac{1}{2}\mathrm{Re}[\mathrm{tr}(U_\beta)], \tag{2.8}$$

and to obtain the distributions $P(T)$ by averaging the above formula over an appropriate ensemble of $M \times M$ unitary matrices U_β.

III. EXPECTATION VALUES

Since $\langle \mathrm{tr}(U_\beta) \rangle = 0$ for any β one obtains

$$\langle T \rangle_\beta = \frac{M}{2}. \tag{3.1}$$

Let $\mathrm{tr}(U_\beta) = z = a e^{i\vartheta}$. In a recent work [13] the following average was derived for canonical circular ensembles: $\langle a^2 \rangle_\beta = 2M/[2+\beta(M-1)]$. Because the distribution of phases ϑ was shown to be uniform, the variances of both parts are equal $\mathrm{var}[\mathrm{Re}(z)] = \mathrm{var}[\mathrm{Im}(z)] = \mathrm{var}(a)/2$. Using Eq. (2.8) we get directly

$$\mathrm{var}(T)_\beta = \langle (T - \langle T \rangle_\beta)^2 \rangle = \frac{M}{8+4\beta(M-1)}. \tag{3.2}$$

Observe that for $\beta=2$ the variance equals 1/8, independently of the matrix size M, while for $\beta=1$ one has var $(T)_1 = M/4(M+1)$, in accordance with earlier results [9]. The variance growths with a decreasing β and tends to $M/8$ in the limiting case $\beta \to 0$.

IV. DISTRIBUTION OF TRANSMISSION COEFFICIENT $P_\beta(T)$

A. The case $M=1$

For $M=1$ the "one-dimensional matrix" $U = e^{i\varphi_1}$ and the phase φ_1 is uniformly distributed in $[0,2\pi)$ for any ensemble. The variable $t = \cos(\varphi_1)$ thus has the distribution $P_t(t) = 1/[\pi\sqrt{(1-t^2)}]$. The transmission coefficient, equal in this case to $T = 1/2 - t/2$, is therefore distributed according to

$$P_{\beta,1}(T) = \frac{1}{\pi\sqrt{T(1-T)}} \tag{4.1}$$

for any value of β.

B. The case $M=2$

Let us start deriving the distribution $P_\beta(a)$ of the absolute value of trace $a = |\mathrm{tr}(U)|$. For $M=2$ the JPD (2.6) reads

$$P_\beta[\varphi_1, \varphi_2] = C_{\beta,2}(\sin\phi)^\beta, \tag{4.2}$$

where $\phi = (\varphi_1 - \varphi_2)/2$. The module of the trace $a = |1 + e^{i2\phi}| = 2\cos\phi$, so employing Eq. (4.2) we obtain the required distribution

$$P_\beta(a) = c_\beta (4 - a^2)^{(\beta-1)/2}, \tag{4.3}$$

where $a \in [0,2]$ and the normalization constant reads

$$c_\beta = \frac{2^{1-\beta}\Gamma\left(\frac{\beta+2}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\beta+1}{2}\right)}.$$

In particular, this distribution is flat for COE, while for CUE one gets a semicircle and recovers the result already mentioned in [13]. According to Eq. (2.8) the transmission coefficient equals $T = 1 - (a\cos\vartheta)/2$. Due to rotational symmetry of the ensembles the distribution of phase ϑ is uniform in $[0,2\pi)$, so the distribution of $t = \cos\vartheta$ is $P_t(t) = 1/[\pi\sqrt{(1-t^2)}]$. Denoting $x = at$ and using Eq. (4.3) one writes an integral for the distribution of x

$$P_\beta(x) = \frac{c_\beta}{\pi}\int_x^2 \frac{(4-a^2)^{(\beta-1)/2}}{\sqrt{a^2 - x^2}}\,da, \tag{4.4}$$

which can be computed numerically in the general case. By a linear change of variables $T = 1 - x/2$ provides the required distribution of transmission coefficient $P_\beta(T)$. Moreover, for most interesting integer values of β the above integral can be evaluated analytically giving

$$P_{\beta,2}(T) = \begin{cases} \frac{2}{\pi^2} K[\sqrt{T(2-T)}] & \text{for } \beta=0 \ \text{(CPE)}, \\ \frac{1}{\pi}\ln\frac{1+\sqrt{T(2-T)}}{|1-T|} & \text{for } \beta=1 \ \text{(COE)}, \\ \frac{4}{\pi^2} T(2-T) D[\sqrt{T(2-T)}] & \text{for } \beta=2 \ \text{(CUE)}, \end{cases} \tag{4.5}$$

where K and D stand for a complete elliptic function of the first and the third kind, respectively, [14]. A somewhat more complicated derivation of this formula, in the case of COE and CUE, has already been given in [9], where the

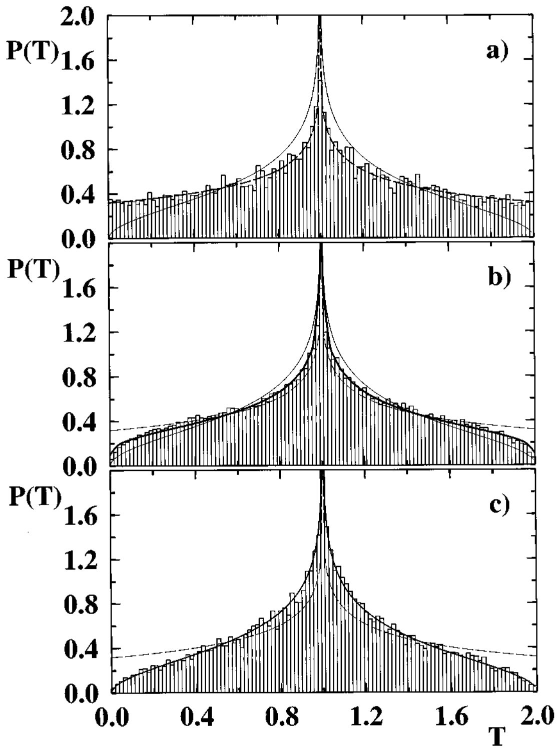

FIG. 1. Distribution of the transmission coefficient $P(T)$ for Poisson-orthogonal transition for $M=2$ obtained of 10^5 random matrices. The control parameter δ is equal to (a) 0.0 (CPE), (b) 0.4, and (c) 1.0 (COE). Solid and dashed narrow lines represent COE and CPE results, respectively, while the bold line in (b) stands for the best fit with $\beta=0.38$.

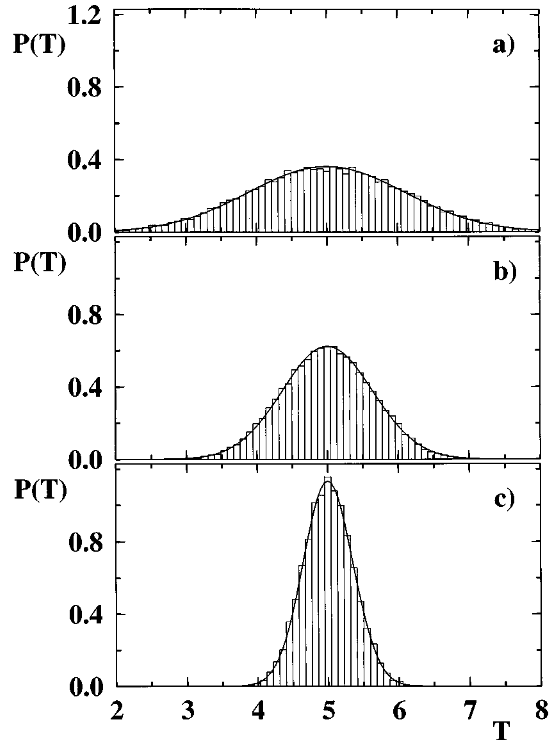

FIG. 2. As in Fig. 1 for Poisson-unitary transition and $M=10$. Solid lines represent the Gaussian distribution with $\langle T\rangle = M/2$ and the variance given by Eq. (3.2) for $\beta=0$ and $\beta=2$ and found numerically for $\beta=0.4$.

elliptic function was expressed by the hypergeometric function as $D(k)=(\pi/4)F(\frac{1}{2},\frac{3}{2};2;k^2)$.

We have generated numerically random unitary matrices of interpolating ensembles using the method proposed in [10]. Figure 1 shows the distribution of transmission coefficients $P(T)$ for $M=2$ and the Poisson-COE transition for three different values of the control parameter δ. Two narrow lines represent the formula (4.5) with $\beta=0$ and $\beta=1$, characteristic for the limiting cases of CPE and COE. The bold line in Fig. 1(b), obtained in the interpolating case, represents the best fit of Eq. (4.4) with $\beta=0.38$ (a coincidence between the values of δ and β is accidental). Good quality of the fit reveals a certain validity of this formula with noninteger values of β for ensembles in between the usual universality classes.

C. Distribution $P_\beta(T)$ for large M

Since the transmission coefficient T is a function of M random variables with finite variances, one expects its distribution to be Gaussian for large M. This fact can be proved rigorously for CUE. For this case the traces $z=\mathrm{tr}(U)$ are distributed (in the limit $M\to\infty$) as an isotropic complex Gaussian variable [13], which guarantees the Gaussian distribution for $x=\mathrm{Re}(z)$, and for $T=(M-x)/2$. The Gaussian property also holds in the Poissonian case, for which x is the sum of M independent terms, each being a cosine of uniformly distributed random phases.

For $M\gg 1$ we conjecture the Gaussian distribution of $P(T)$ for any value of β with the mean equal to $M/2$ and the variance given by Eq. (3.2). Numerical tests revealed a Gaussian character of this distribution for any β already for $M=10$. Figure 2 presents the distribution of transmission coefficient for the Poisson-CUE transition and three values of the control parameter δ.

V. CONCLUDING REMARKS

Composed ensembles of random unitary matrices were applied for analysis of the transmission coefficient in symmetric chaotic systems described by S matrices with block symmetries. We simplified the derivation of distribution $P_\beta(T)$ presented by Gopar *et al.* [9] and generalized their results proposing a family of interpolating distributions.

It should be noted that the Poissonian case $\beta=0$ discussed above is not capable of describing an effect of localization. It corresponds rather to the case of scattering on a half transparent mirror, in which each scattering mode acquires a random phase shift. Another generalization of the model designed to describe effects of localization is a subject of a subsequent publication [15].

ACKNOWLEDGMENTS

It is a pleasure to thank Marek Kuś, Marcin Poźniak, Petr Šeba, and Jakub Zakrzewski for fruitful discussions. Financial support by Komitet Badań Naukowych via Grant No. 2P03B 03810 is gratefully acknowledged.

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